Section 1.2 Row Reduction and Echelon Forms

Echelon Form and Reduce Echelon Form

Definition

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros. so all rows of only 0's are at the bottom
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Example 1 Determine which matrices are in reduced echelon form and which others are only in echelon form.

a. $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ → Reduced echelon form

b. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ Not echelon. By checking R1 and R2, the matrix does not satisfy #2 and 3

d. $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Echelon form. But not reduced echelon form.

> **Theorem 1 (Uniqueness of the Reduced Echelon Form)**
>
> Each matrix is row equivalent to one and only one reduced echelon matrix.

🤔 **Question:** Given a matrix A, how to produce a (reduced) echelon matrix that is row equivalent to A? We will answer this question in the rest part of this section. (Row Reduction Algorithm)

Pivot Positions

Definition

- A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.
- A **pivot column** is a column of A that contains a pivot position.

Example 2 Row reduce the given matrix to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

$$A = \begin{array}{c} R1 \\ R2 \\ R3 \\ R4 \end{array} \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow[\text{to put 1 in the pivot position}]{\text{interchange } R1 \leftrightarrow R4} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

pivot

pivot column

next pivot

R1 added to R2

2xR1 added to R3

to create 0's below the pivot. Choose 2 as the next pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

next pivot column.

$-\frac{5}{2} \times R2$ added to R3

$\frac{3}{2} \times R2$ added to R4

to create 0's below the pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

interchange R3 $\leftrightarrow$ R4

to produce a leading entry in column 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note it is in echelon form. We process the following steps to get the reduce echelon form

$$\xrightarrow[R3 \to R3/(-5)]{R2 \to R2/2} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot

pivot

pivot

$$\xrightarrow[R1 \to 9 \times R1 + R1]{R2 \to 3 \times R3 + R2} \begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1 \to (-4) \times R2 + R1} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot positions

pivot columns

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

columns 1, 2, 4 are pivot columns

With **Example 2**, we are ready to describe an efficient procedure for transforming a matrix into an echelon or reduced echelon matrix:

The Row Reduction Algorithm Step 1 - 4 are called forward phase

STEP 1: Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

STEP 2: Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

STEP 3: Use row replacement operations to create zeros in all positions below the pivot.

STEP 4: Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1 to 3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

STEP 5: Backward phase. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Example 3 Using the algorithm above, row reduce the given matrix to reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

pivot column

STEP 1: Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

STEP 2: Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

So we interchange R1 and R3 (You can also interchange R1 and R2)

pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

STEP 3: Use row replacement operations to create zeros in all positions below the pivot.

We have two options: ① Divide R1 by 3 to obtain 1 in the pivot position. Then add $(-3)\times R1$ to R2, or

② Add $(-1)\times R1$ to R2

I'll choose option ② here:

$$\xrightarrow{(-1)\times R1 \text{ added to } R2} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

STEP 4: Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1 to 3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

$$\begin{array}{c} \text{pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \begin{array}{l} \leftarrow \text{cover} \\ \\ \\ \end{array} \\ \text{Next pivot column} \end{array}$$

Again, we have two options: ① Divide R2 by 2 to get 1 in the pivot position. Then add $(-3)\times R2$ to $R3$

or, ② Add $\left(-\frac{3}{2}\right)\times R2$ to $R3$.

We choose ② in our calculation:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

pivot, no more rows to modify.

So we are done with Step 4. The above matrix is an echelon form (You can double check this with the 3 properties on Page 1)

If we want the reduced echelon form, we continue with Step 5:

STEP 5: Backward phase. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

$$\begin{bmatrix}3 & -9 & 12 & -9 & 6 & 15\\ 0 & 2 & -4 & 4 & 2 & -6\\ 0 & 0 & 0 & 0 & 1 & 4\end{bmatrix}$$

right most pivot

$$\xrightarrow[R1+(-6)\times R3]{R2+(-2)\times R3} \begin{bmatrix}3 & -9 & 12 & -9 & 0 & -9\\ 0 & 2 & -4 & 4 & 0 & -14\\ 0 & 0 & 0 & 0 & 1 & 4\end{bmatrix}$$

to create 0's above the pivot

next right most pivot

$$\xrightarrow{R2/2} \begin{bmatrix}3 & -9 & 12 & -9 & 0 & -9\\ 0 & 1 & -2 & 2 & 0 & -7\\ 0 & 0 & 0 & 0 & 1 & 4\end{bmatrix}$$

to make the pivot 1

$$\xrightarrow{R1+9\times R2} \begin{bmatrix}3 & 0 & -6 & 9 & 0 & -72\\ 0 & 1 & -2 & 2 & 0 & -7\\ 0 & 0 & 0 & 0 & 1 & 4\end{bmatrix}$$

$$\xrightarrow{R1/3} \begin{bmatrix}1 & 0 & -2 & 3 & 0 & -24\\ 0 & 1 & -2 & 2 & 0 & -7\\ 0 & 0 & 0 & 0 & 1 & 4\end{bmatrix}$$

This is the reduced echelon form of the oringinal matrix

Solutions of Linear Systems

Definitions

Basic variable: the variables corresponding to pivot columns in the matrix

Free variable: the other variables

Example 3 The augmented matrix of a linear system has been written in the reduced echelon form as follows

Notice: The system has 3 eqns with 3 varibles by checking the size of the matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Column 1 and 2 are the pivot columns.

(1) Determine the basic variable and free variable.

Basic varible: x_1, x_2

Free variable: x_3

(2) Find the general solution of the above system.

ANS: The corresponding system is

$$\begin{cases} x_1 \quad - 5x_3 = 1 \\ x_2 + x_3 = 4 \\ 0 = 0 \end{cases}$$

So the solution can be writte as

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

Note: Each different choice of x_3 determines a solution to the system, and every solution to the system is determined by a choice of x_3

Existence and Uniqueness Questions

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column-that is, if and only if an echelon form of the augmented matrix has no row of the form

$$[0 \quad \cdots \quad 0 \quad b] \quad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either

(i) a unique solution, when there are no free variables, or

(ii) infinitely many solutions, when there is at least one free variable.

Example 4 Choose h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

$x_1 + 3x_2 = 2$

$3x_1 + hx_2 = k$

ANS: The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & h & k \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{bmatrix}$$

(a) By the Thm above (blue box), the system has no solution if and only if R2 is of the form $[0 \quad 0 \quad b]$ $b \neq 0$

i.e., $\begin{cases} h-9=0 \\ k-6 \neq 0. \end{cases} \Rightarrow \begin{cases} h=9 \\ k \neq 6. \end{cases}$

Thus the system has no solution when $h=9$ and $k \neq 6$.

(b) By (i) in the Thm, the system has unique solution when there is no free varible. So $h-9 \neq 0$ (i.e. it is a pivot). Thus when $h \neq 9$, the system has unique solution

(c) By (ii) in the Thm, the system has many solutions when there is a free varible. This means $h-9=0$, $k-6=0$, i.e. $h=9$ and $k=6$.

The following procedure outlines how to find and describe all solutions of a linear system.

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.

2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.

3. Continue row reduction to obtain the reduced echelon form.

4. Write the system of equations corresponding to the matrix obtained in step 3 .

5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

The following question is left as an exercise. I will provide the complete notes for solving them after the lecture.

Exercise 5 Find the general solutions of the systems whose augmented matrices are given as follows.

(1) $\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 11 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

ANS:

(1) $\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & -1 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & 1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 3 \end{bmatrix}$

The corresponding system of equations:

$$x_1 = -5$$
$$x_2 = 3$$

The basic variables (corresponding to the pivot positions) are x_1 and x_2. The remaining variable x_3 is free. Solve for the basic variables in terms of the free variable. In this particular problem, the basic variables do not depend on the value of the free variable.

General solution: $\begin{cases} x_1 = -5 \\ x_2 = 3 \\ x_3 \text{ is free} \end{cases}$

Remark: A common error in Exercise 5 (1) is to assume that x_3 is zero. To avoid this, identify the basic variables first. Any remaining variables are free. (This type of computation will arise in Chapter 5.)

(2) $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

The corresponding system of equations:

$$\begin{aligned} x_1 - 7x_2 + 6x_4 &= 5 \\ x_3 - 2x_4 &= -3 \\ 0 &= 0 \end{aligned}$$

Columns 1 and 3 are pivot columns. So

Basic variables: x_1 and x_3; free variables: x_2, x_4. General solution: $\begin{cases} x_1 = 5 + 7x_2 - 6x_4 \\ x_2 \text{ is free} \\ x_3 = -3 + 2x_4 \\ x_4 \text{ is free} \end{cases}$

(3) $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 0 & 9 & -6 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & -3 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

The corresponding system of equations:

$$\begin{aligned} x_1 \qquad\qquad - 3x_5 &= -3 \\ x_2 \qquad - 4x_5 &= 1 \\ x_4 + 9x_5 &= -4 \\ 0 &= 0 \end{aligned}$$

Columns 1, 2, 4 are pivot columns. So

Basic variables: x_1, x_2, x_4; free variables: x_3, x_5. General solution: $\begin{cases} x_1 = -3 + 3x_5 \\ x_2 = 1 + 4x_5 \\ x_3 \text{ is free} \\ x_4 = -4 - 9x_5 \\ x_5 \text{ is free} \end{cases}$

Remark: The *Study Guide* discusses the common mistake $x_3 = 0$.